4 Parametric Representation of Curves; Lines in $\mathbb{R}^3$

IN THIS SECTION parametric equations, parameterizing a curve, lines in $\mathbb{R}^3$

PARAMETRIC EQUATIONS

Until now, most of the curves we have considered have been graphs of functions of the form $y = f(x)$. This form of representation is limited by the requirements of the vertical line test, which precludes closed curves such as circles or any curve with a self-intersection. However, such restrictions are often not necessary for curves where the coordinates x and y of each point $P(x, y)$ on the curve are themselves functions of a third variable, called a *parameter*. We will begin by examining parametric representations in $\mathbb{R}^2$, and then extend our results to $\mathbb{R}^3$.

Parametric Representation of a Curve in $\mathbb{R}^2$

Let f and g be continuous functions of t on an interval I; then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are called **parametric equations** with **parameter** t. As t varies over the **parametric set** I, the points $(x, y) = (f(t), g(t))$ trace out a **parametric curve**.

Note: The letter "t" used for the parameter does not necessarily denote time, although in many applications, time is a suitable parameter. Indeed, any letter or symbol may be used to denote a parameter.

EXAMPLE 1 Sketching the path of a parametric curve

Sketch the path of the curve $x = t^2 - 9$, $y = \frac{1}{3}t$ for $-3 \le t \le 2$.

Solution

Values of x and y corresponding to various choices of the parameter t are shown in the following table:

t	x	y	
-3	0	-1	(Starting or **initial** point)
-2	-5	$-\frac{2}{3}$	
-1	-8	$-\frac{1}{3}$	
0	-9	0	
1	-8	$\frac{1}{3}$	
2	-5	$\frac{2}{3}$	(Ending or **terminal** point)

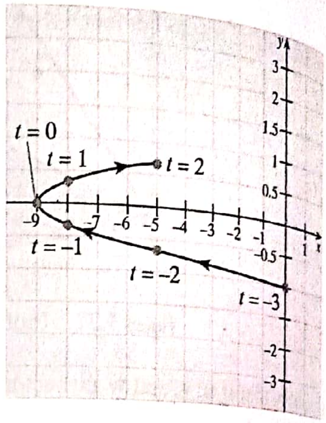

Figure 9.33 Graph of $x = t^2 - 9$, $y = \frac{1}{3}t$, for $-3 \le t \le 2$. Notice how the arrows show the orientation as t increases from -3 to 2.

The graph is shown in Figure 9.33. ∎

If you are using a computer or a graphing calculator, plotting points can be an effective way of sketching a parametric curve. Sometimes, however, we wish to eliminate the parameter to obtain a Cartesian equation. For instance, in Example 1, we have $y = \frac{1}{3}t$, so $t = 3y$, and by substituting into the equation $x = t^2 - 9$, we obtain

$$x = (3y)^2 - 9 = 9y^2 - 9$$

which is the Cartesian equation for a parabola that opens to the right. Because of the domain of the parameter t, we see that the parametric curve in Figure 9.33 is a subset of the set of points that satisfy the equation $x = 9y^2 - 9$.

Parameterizations are not unique. For example, the curve with parametric equations

$$x = 9(9t^2 - 1), \quad y = 3t \quad \text{for } -\tfrac{1}{3} \le t \le \tfrac{2}{9}$$

is the same as the curve in Figure 9.33.

EXAMPLE 2 Sketching the path of a parametric curve by eliminating the parameter

Describe the path $x = \sin \pi t$, $y = \cos 2\pi t$ for $0 \le t \le 0.5$.

Solution

Using a double angle identity, we find

$$\cos 2\pi t = 1 - 2\sin^2 \pi t$$

so that

$$y = 1 - 2x^2$$

We recognize this as a Cartesian equation for a parabola. Because $y' = -4x$, we can find the critical number $x = 0$, which locates the vertex of the parabola at $(0, 1)$. The parabola is the curve shown in color as the dashed curve in Figure 9.34.

Because t is restricted to the interval $0 \le t \le 0.5$, the parametric representation involves only part of the right side of the parabola $y = 1 - 2x^2$. The curve is oriented from the point $(0, 1)$, where $t = 0$, to the point $(1, -1)$, where $t = 0.5$, and is the portion of the parabola shown in black in Figure 9.34. ■

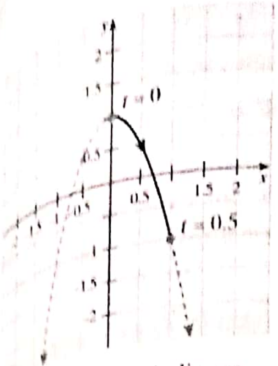

Figure 9.34 The parabolic arc $x = \sin \pi t$, $y = \cos 2\pi t$ for $0 \le t \le 0.5$

When it is difficult to eliminate the parameter from a given parametric representation, we can sometimes get a good picture of the parametric curve by plotting points.

EXAMPLE 3 Describing a spiraling path

Discuss the path of the curve described by the parametric equations

$$x = e^{-t}\cos t, \qquad y = e^{-t}\sin t \qquad \text{for } t \ge 0$$

Solution

We have no convenient way of eliminating the parameter so we write out a table of values (x, y) that correspond to various values of t. The curve is obtained by plotting these points in a Cartesian plane and passing a smooth curve through the plotted points, as shown in Figure 9.35.

t	x	y
0	1	0
$\frac{\pi}{4}$	0.32	0.32
1	0.20	0.31
$\frac{\pi}{2}$	0	0.21
2	−0.06	0.12
π	−0.04	0
$\frac{3\pi}{2}$	0	−0.01
2π	0.00	0

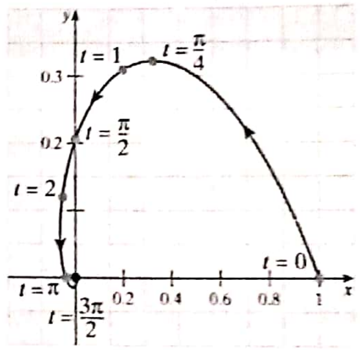

Figure 9.35 Graph of $x = e^{-t}\cos t$, $y = e^{-t}\sin t$ for $t \ge 0$

Note that for each value of t, the distance from $P(x, y)$ on the curve to the origin is

$$\sqrt{x^2 + y^2} = \sqrt{(e^{-t}\cos t)^2 + (e^{-t}\sin t)^2} = \sqrt{e^{-2t}(1)} = e^{-t}$$

Because e^{-t} decreases as t increases, it follows that P gets closer and closer to the origin as t increases. However, because $\cos t$ and $\sin t$ vary between -1 and $+1$, the approach is not direct but takes place along a spiral. ■

PARAMETERIZING A CURVE

So far, our examples have dealt with sketching a parametric curve given the parametric equations. In general, this process may be tedious, but generally can be done easily using technology. However, the reverse process, finding a suitable set of parametric equations for a given curve, is an art for which there is no simple procedure. Indeed, a given curve can have many different parameterizations and there are curves for which no simple parameterization can be given. The following two examples illustrate various methods for parameterizing a given curve.

EXAMPLE 4 Parameterizing two curves

In each of the following cases, parameterize the given curve:

a. $y = 9x^2$ **b.** $r = 5\cos^3\theta$ in polar coordinates

Solution

a. The usual parameterization for a parabola is to let the parameter t be the variable that is squared: $x = t$, $y = 9t^2$. However, another parameterization is to let $t = 3x$, so that $x = \frac{1}{3}t$ and $y = t^2$.

b. In polar coordinates we have $x = r\cos\theta$, $y = r\sin\theta$, so we can parameterize x and y in terms of the parameter θ:

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
$$= (5\cos^3\theta)\cos\theta \qquad\qquad = (5\cos^3\theta)\sin\theta$$
$$= 5\cos^4\theta$$

EXAMPLE 5 Modeling Problem: Finding parametric equations for a trochoid

A bicycle wheel has radius a and a reflector is attached at a point P on a spoke of a bicycle wheel at a fixed distance d from the center. Find parametric equations for the curve described by P as the wheel rolls along a straight line without slipping. Such a curve is called a **trochoid**, and is shown in Figure 9.36.

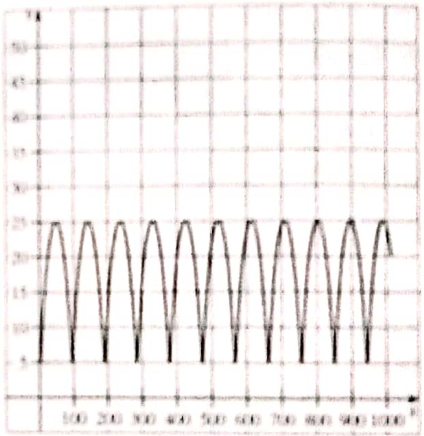

Figure 9.36 The path of a reflector placed 5 in. from the tire on the wheel of a bike with 30-in. wheel. This is an example of a trochoid.

Solution

Assume that the wheel rolls along the x-axis and that the center C of the wheel begins at $(0, a)$ on the y-axis. Further assume that P also starts on the y-axis, d units below C. Figure 9.37 shows the initial position of the wheel and its position after turning through an angle θ (radians).

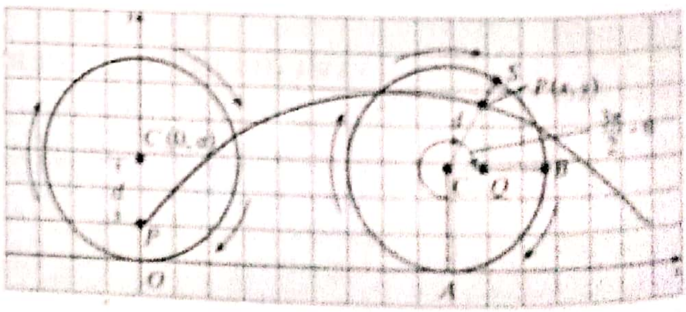

Figure 9.37 The path of a reflector on a bicycle

We begin by labeling some points: The point A is on the x-axis directly beneath C, whereas B is the point where the horizontal line through C meets the rim of the wheel. Finally, Q is the point on BC directly beneath P, and S is the point where the line through C and P intersects the rim. Let P have coordinates (x, y).

We need to find representations (in terms of a, d, and θ) for x and y.

$$\begin{aligned} x &= |\overline{OA}| + |\overline{CQ}| \\ &= a\theta + |\overline{CQ}| \end{aligned}$$

Bacause the wheel rolls along the x-axis without slipping, $|\overline{OA}|$ is the same as the arc length from A to S, so $|\overline{OA}| = a\theta$.

$$\begin{aligned} y &= |\overline{AC}| + |\overline{QP}| \\ &= a + |\overline{QP}| \end{aligned}$$

To complete our evaluation of x and y, we need to compute $|\overline{CQ}$ and $|\overline{QP}|$. These are sides of ΔPCQ. Note that $\angle PCQ = \frac{3\pi}{2} - \theta$; therefore, by the definitions of cosine and sine, we have

$$\cos\left(\frac{3\pi}{2} - \theta\right) = \frac{|\overline{CQ}|}{d} \quad \text{so} \quad |\overline{CQ}| = d\cos\left(\frac{3\pi}{2} - \theta\right) = -d\sin\theta$$

Similarly, $|\overline{QP}| = d\sin\left(\frac{3\pi}{2} - \theta\right) = -d\cos\theta$.

We can now substitute these values for $|\overline{CQ}|$ and $|\overline{QP}|$ into the equations we derived for x and y.

$$\begin{aligned} x &= a\theta + |\overline{CQ}| = a\theta - d\sin\theta \\ y &= a + |\overline{QP}| = a - d\cos\theta \end{aligned}$$

■

The special case where P is on the rim of the wheel in Example 5 (when $d = a$) is a curve called a **cycloid**. There are several problems involving these and similar curves in the problem set.

If f, g, and h are continuous functions of a variable t on an interval I, then

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

are parametric equations with parameter t, and as t varies over I, the points $(x, y, z) = (f(t), g(t), h(t))$ trace out a parametric curve in $\mathbb{R}^3$. The rest of this section is devoted to examining lines in $\mathbb{R}^3$, which we will represent in parametric form. More general curves in $\mathbb{R}^3$ will be studied in Chapter 10.

LINES IN $\mathbb{R}^3$

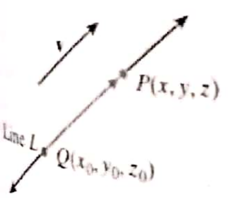

Figure 9.38 If L is parallel to $\mathbf{v}$ and contains Q, then P is on L whenever $QP = t\mathbf{v}$.

As in the plane, a line in $\mathbb{R}^3$ is completely determined once we know one of its points and its direction. We used the concept of slope to measure the direction of a line in the plane, but in space, it is more convenient to specify direction with vectors.

Suppose L is a line in $\mathbb{R}^3$ that is parallel to the vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ and also contains $Q(x_0, y_0, z_0)$, as shown in Figure 9.38. We say that the line has **direction numbers** A, B, and C and denote these direction numbers by $[A, B, C]$. The vector $\mathbf{v}$ is called a **direction vector** of the line L. If $P(x, y, z)$ is any point on L, then the vector $\mathbf{QP}$ is parallel to $\mathbf{v}$ and must satisfy the vector equation $\mathbf{QP} = t\mathbf{v}$ for some number t. If we introduce coordinates and use the standard representation, we can rewrite this vector equation as

9.4 PROBLEM SET

1. **WHAT DOES THIS SAY?** What is a parameter?
2. **WHAT DOES THIS SAY?** Contrast the parametric and symmetric forms of the equation of a line.

Find an explicit relationship between x and y in Problems 3–16 by eliminating the parameter. In each case, sketch the path described by the parametric equations over the prescribed interval.

3. $x = t + 1, y = t - 1, 0 \leq t \leq 2$
4. $x = -t, y = 3 - 2t, 0 \leq t \leq 1$
5. $x = 60t, y = 80t - 16t^2, 0 \leq t \leq 3$
6. $x = 30t, y = 60t - 9t^2, -1 \leq t \leq 2$
7. $x = t^3, y = t^2, t \geq 0$
8. $x = t^4, y = t^2, -1 \leq t \leq \sqrt{2}$
9. $x = 3 \cos \theta, y = 3 \sin \theta, 0 \leq \theta \leq 2\pi$
10. $x = 2 \sin \theta, y = 2 \cos \theta, 0 \leq \theta \leq 2\pi$
11. $x = 1 + \sin t, y = -2 + \cos t, 0 \leq t \leq 2\pi$
12. $x = 1 + \sin^2 t, y = -2 + \cos t, 0 \leq t \leq \pi$
13. $x = 4 \tan 2t, y = 3 \sec 2t, 0 \leq t \leq \pi$

$$\frac{x-1}{2}=\frac{y+2}{4}=\frac{z+5}{1}$$

14. $x = 4\sec 2t,\ y = 2\tan 2t,\ 0 \le t \le \pi$
15. $x = t^3,\ y = 3\ln t,\ t > 0$
16. $x = e^t,\ y = e^{-t},\ (-\infty, \infty)$

Find the parametric and symmetric equations for the line(s) passing through the given points with the properties described in Problems 17–25.

17. $(1, -1, -2)$; parallel to $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$
18. $(1, 0, -1)$; parallel to $3\mathbf{i} + 4\mathbf{j}$
19. $(1, -1, 2)$; through $(2, 1, 3)$
20. $(2, 2, 3)$; through $(1, 3, -1)$
21. $(1, -3, 6)$; parallel to $\dfrac{x-5}{1} = \dfrac{y+2}{-3} = \dfrac{z}{-5}$
22. $(1, -1, 2)$; parallel to $\dfrac{x+3}{4} = \dfrac{y-2}{5} = \dfrac{z+5}{1}$
23. $(0, 4, -3)$; parallel to $\dfrac{x-1}{22} = \dfrac{y+2}{-6} = \dfrac{z-1}{10}$
24. $(1, 0, -4)$; parallel to $x = -2 + 3t,\ y = 4 + t,\ z = 2 + 2t$
25. $(-1, 1, 6)$; perpendicular to $3x + y - 2z = 5$
26. Find the parametric form of the equation of the line passing through $(3, -1, 0)$ parallel to both the xy- and yz-planes.

Find the points of intersection of each line in Problems 27–30 with each of the coordinate planes.

27. $\dfrac{x-4}{4} = \dfrac{y+3}{3} = \dfrac{z+2}{1}$
28. $\dfrac{x+1}{1} = \dfrac{y+2}{2} = \dfrac{z-6}{3}$
29. $x = 6 - 2t,\ y = 1 + t,\ z = 3t$
30. $x = 6 + 3t,\ y = 2 - t,\ z = 2t$

In Problems 31–36, tell whether the two lines intersect, are parallel, are skew, or coincide. If they intersect, give the point of intersection.

31. $\dfrac{x-4}{2} = \dfrac{y-6}{-3} = \dfrac{z+2}{5};\ \dfrac{x}{4} = \dfrac{y+2}{-6} = \dfrac{z-3}{10}$
32. $x = 4 - 2t,\ y = 6t,\ z = 7 - 4t;$
$x = 5 + t,\ y = 1 - 3t,\ z = -3 + 2t$
33. $x = 3 + 3t,\ y = 1 - 4t,\ z = -4 - 7t;$
$x = 2 + 3t,\ y = 5 - 4t,\ z = 3 - 7t$
34. $x = 2 - 4t,\ y = 1 + t,\ z = \frac{1}{2} + 5t;$
$x = 3t,\ y = -2 - t,\ z = 4 - 2t$
35. $\dfrac{x-3}{2} = \dfrac{y-1}{-1} = \dfrac{z-4}{1};\ \dfrac{x+2}{3} = \dfrac{y-3}{-1} = \dfrac{z-2}{1}$
36. $\dfrac{x+1}{2} = \dfrac{y-3}{-1} = \dfrac{z-2}{1};\ \dfrac{x+1}{2} = \dfrac{y+1}{3} = \dfrac{z-3}{-4}$
37. Find two unit vectors parallel to the line
$$\frac{x-3}{4} = \frac{y-1}{2} = \frac{z+1}{1}$$
38. Find two unit vectors parallel to the line

B *Find the parametric equations for each of the curves in Problems 39–44.*

39. A circle of radius 3, centered at the origin, oriented counterclockwise.
40. A circle of radius 2 centered at the origin, oriented clockwise.
41. The ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$, oriented counterclockwise.
42. The parabola $y^2 = 4x + 9$, oriented from $(4, -5)$ to $(0, 3)$
43. The hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.
44. The ellipse $\dfrac{(x-2)^2}{3} + \dfrac{(y+3)^2}{5} = 1$.
45. Describe the path of the curve described by $x = \sin \pi t$, $y = \cos 2\pi t$ for $0 \le t < 1$.
46. Let $x = 4a \sin t$, $y = b\cos^2 t$. Express y as a function of x.
47. Show that the vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ is orthogonal to the line that passes through the points $P(0, 0, 1)$ and $Q(2, 1, -1)$.
48. Show that the vector $\mathbf{v} = 7\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$ is orthogonal to the line passing through the points $P(-2, 2, 7)$ and $Q(3, -3, 2)$.
49. Find constants a and b so that the following lines coincide:
$$L_1: \frac{x-a}{2} = \frac{y-1}{4} = \frac{z+2}{1}$$
and
$$L_2: \frac{x-2}{-4} = \frac{y-b}{-8} = \frac{z+1}{-2}$$
50. Consider the lines
$$L_1: x = -1 + 2t, \quad y = 3 - t, \quad z = 2 + 2t$$
$$L_2: x = -2 - t, \quad y = 5 + 2t, \quad z = -2t$$
Show that L_1 and L_2 intersect and find the acute angle θ (to the nearest degree) between them.
51. Find an equation for the line L_1 that contains the point $P(-3, 1)$ and is orthogonal to the line
$$L_1: \quad x = 2 - t, \quad y = 1 - 2t, \quad z = 5 + t$$

C 52. What can be said about the lines
$$\frac{x - x_0}{a_1} = \frac{y - y_0}{b_1} = \frac{z - z_0}{c_1}$$
and
$$\frac{x - x_0}{a_2} = \frac{y - y_0}{b_2} = \frac{z - z_0}{c_2}$$
in the case where $a_1a_2 + b_1b_2 + c_1c_2 = 0$?

53. **Modeling Problem** A circle of radius R rolls without slipping on the *outside* of a fixed circle of radius a. Assume the fixed circle is centered at the origin and that the moving point begins at $(a,0)$. Let t be the angle measured from the positive x-axis to the ray from the origin to the center of the rolling circle. Show that this *epicycloid* may be modeled by the parametric equations
$$x = (a + R)\cos t - R\cos\left(\frac{a+R}{R}\right)t,$$

$$y = (a + R)\sin t - R\sin\left(\frac{a+R}{R}\right)t$$

Modeling Problem A circle of radius R rolls without slipping on the inside of a fixed circle of radius a. Find parametric equations to model the curve traced out by a point P on the circumference of the rolling circle of radius R. Let t be the angle measured from the positive x-axis to the ray that passes through the center of the rolling circle, and assume that the point P

begins on the x-axis (that is, P has coordinates $(a, 0)$ when $t = 0$). Show that this *hypocycloid* has parametric equations

$$x = (a - R)\cos t + R\cos\left(\frac{a-R}{R}\right)t,$$

$$y = (a - R)\sin t - R\sin\left(\frac{a-R}{R}\right)t$$

in $\mathbb{R}^3$